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*Published in:*

Proceedings of 22nd International Congress on Sound and Vibration

*Publication date:*

2015

*Document Version*

Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*

Sorokin, V., & Thomsen, J. J. (2015). Effects of weak nonlinearity on dispersion relations and frequency band-gaps of periodic structures. In M. J. Crocker, M. Pawelczyk, F. Pedrielli, E. Carletti, & S. Luzzi (Eds.), *Proceedings of 22nd International Congress on Sound and Vibration: Major challenges in Acoustics, Noise and Vibration Research, 2015* [989 (I)] The International Institute of Acoustics and Vibration (IIAV).

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# EFFECTS OF WEAK NONLINEARITY ON DISPERSION RELATIONS AND FREQUENCY BAND-GAPS OF PERIODIC STRUCTURES

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The analysis of the behaviour of linear periodic structures can be traced back over 300 years, to Sir Isaac Newton, and still attracts much attention. An essential feature of periodic structures is the presence of frequency band-gaps, i.e. frequency ranges in which waves cannot propagate. Determination of band-gaps and the corresponding attenuation levels is an important practical problem. Most existing analytical methods in the field are based on Floquet theory; e.g. this holds for the classical Hill's method of infinite determinants, and the method of space-harmonics. However, application of these for nonlinear problems is impossible or cumbersome, since Floquet theory is applicable for linear systems only. Thus the nonlinear effects for periodic structures are not yet fully uncovered, while at the same time applications may demand effects of nonlinearity on structural response to be accounted for. The present work deals with analytically predicting dynamic responses for nonlinear continuous elastic periodic structures. Specifically, the effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations are analyzed. Modulation of the beam structural properties is not required to be small or piecewise constant. Various sources of nonlinearity are analyzed, namely, nonlinear (true) curvature, nonlinear inertia due to longitudinal motions of the beam, nonlinear material, and also nonlinearity associated with mid-plane stretching. A novel approach, the Method of Varying Amplitudes, is employed. This implies representing a solution in the form of a harmonic series with varying amplitudes; however, in contrast to averaging methods, the amplitudes are not required to vary slowly in space. As a result, a shift of band-gaps to a higher (or lower) frequency range is revealed, while the width of the band-gaps appears relatively insensitive to (weak) nonlinearity. The results are validated by numerical simulation, and explanations of the effects are suggested.

## 1. Introduction

Possible sources of nonlinearities for a Bernoulli-Euler beam performing bending oscillations were discussed in many works, see e.g., the classical monograph [1], and the paper [2]. In [1] the main sources were identified as the nonlinear stiffness and the nonlinear inertia. It was noted that character of the nonlinearity strongly depends on specific boundary conditions applied to the beam.

For example, when there is no restriction on longitudinal motions of one (or both) of the beam ends, then large deflections are possible, so that the nonlinear (true) curvature and the nonlinear inertia due to longitudinal motions of the beam should be taken into account. Also material nonlinearity may be significant in this case. If both ends of the beam are restricted to move in the longitudinal direction, then another source of nonlinearity becomes most important, namely the mid-plane stretching. This nonlinearity appears to be much stronger than all others [3,4], and influences the beam response already at relatively small deflections.

As is seen, in real structures some of the nonlinearities are imposed due to boundary conditions, so that finite structures should be considered. On the other hand, analysis of dispersion relations and frequency band-gaps usually implies considering infinite structures [5,6]. Transition from infinite to finite structures and discussion of whether it is valid to study dispersion relations and band-gaps for finite structures are given in many papers, see e.g. [5,6]. The basic assumption in such a transition is that the considered structure is sufficiently long, so that waves from band-gaps ranges are attenuated before reaching the boundaries.

In the present paper the effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations are analysed. Modulation of the beam structural properties is not required to be small or piecewise constant, so that even the corresponding linear problem does not allow an exact solution.

A novel approach, the Method of Varying Amplitudes (MVA) [7,8], is employed in the paper. This approach is inspired by the method of direct separation of motions (MDSM) [9], and may be considered a natural continuation of the classical methods of harmonic balance [1] and averaging [10]. It implies representing a solution in the form of a harmonic series with varying amplitudes; however, in contrast to the asymptotic methods, the amplitudes are not required to vary slowly. The approach is strongly related also to Hill's method of infinite determinants [1,5,11], and to the method of space-harmonics [6].

## 2. Governing equations

Consider the first case when there is no restriction on longitudinal motions of the beam, and large transverse deflections are possible. Internal bending moment of a Bernoulli-Euler beam with varying spatial properties is defined by the following expression:

$$(1) \quad M(\tilde{x}, \tilde{t}) = EI(\tilde{x}) \left[ 1 + \frac{1}{2} (\tilde{w}')^2 - \tilde{\beta}_n (\tilde{w}'')^2 \right] \tilde{w}''.$$

Here the effects of the nonlinear (true) curvature and the nonlinear material are taken into account;  $\tilde{w} = \tilde{w}(\tilde{x}, \tilde{t})$  is the transverse displacement of the beam at time  $\tilde{t}$  and axial coordinate  $\tilde{x}$ , primes denote derivatives with respect to  $\tilde{x}$ , and the coefficient  $\tilde{\beta}_n$  defines the nonlinearity of the beam's material stress-strain relation.

The additional longitudinal force due to effects of inertia is [2]:

$$(2) \quad \tilde{N}(\tilde{x}, \tilde{t}) = -\frac{1}{2} \int \rho A(\tilde{x}) \frac{\partial^2}{\partial \tilde{t}^2} \left[ \int (\tilde{w}')^2 d\tilde{x} \right] d\tilde{x}.$$

The beam mass per unit length  $\rho A(\tilde{x})$  and bending stiffness  $EI(\tilde{x})$  are assumed to be periodically varying in the axial coordinate:

$$(3) \quad \rho A(\tilde{x}) = \rho A(\tilde{x} + \Theta), \quad EI(\tilde{x}) = EI(\tilde{x} + \Theta),$$

where  $\Theta$  is period of modulation. Our aim is to reveal the principle effects of nonlinearities on the dispersion relation and frequency band-gaps of a periodic beam. Consequently, expanding  $\rho A(\tilde{x})$

and  $EI(\tilde{x})$  in a trigonometric Fourier series, and taking into account only the primary harmonic, one obtains:

$$(4) \quad \rho A(\tilde{x}) = \rho A_0(1 + \chi_A \sin(k\tilde{x} + \phi)), \quad EI(\tilde{x}) = EI_0(1 + \chi_I \sin(k\tilde{x} + \phi)),$$

where  $0 \leq \chi_A < 1$ ,  $0 \leq \chi_I < 1$ ,  $k = 2\pi / \Theta$ . Note that the modulation amplitudes  $\chi_A$  and  $\chi_I$  are not required to be equal.

Consequently, the governing equation for transverse motions of the beam takes the form:

$$(5) \quad \rho A_0(1 + \chi_A \sin(k\tilde{x} + \phi)) \frac{\partial^2 \tilde{w}}{\partial \tilde{t}^2} - (\tilde{N} \tilde{w}')' + EI_0 \left[ (1 + \chi_I \sin(k\tilde{x} + \phi)) \left( 1 + \frac{1}{2} (\tilde{w}')^2 - \tilde{\beta}_n (\tilde{w}'')^2 \right) \tilde{w}'' \right] = 0.$$

Introducing a new spatial coordinate  $x = k\tilde{x} + \phi$ , timescale  $t = \tilde{\omega} \tilde{t}$ , and the non-dimensional variable  $w = k\tilde{w}$ , where  $\tilde{\omega} = k^2 \sqrt{EI_0 / (\rho A_0)}$  is the frequency of waves of length  $2\pi / k$  propagating in the corresponding homogeneous beam, (5) can be rewritten in dimensionless form:

$$(6) \quad (1 + \chi_A \sin x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ (1 + \chi_I \sin x) \left( 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \beta_n \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right) \frac{\partial^2 w}{\partial x^2} \right] - \frac{\partial}{\partial x} \left( N \frac{\partial w}{\partial x} \right) = 0,$$

where  $\beta_n = \tilde{\beta}_n k^2$ , and

$$(7) \quad N(x, t) = -\frac{1}{2} \int (1 + \chi_A \sin x) \frac{\partial^2}{\partial t^2} \left[ \int \left( \frac{\partial w}{\partial x} \right)^2 dx \right] dx.$$

Note that since  $w = k\tilde{w}$  the effect of the nonlinearities depends not only on the magnitude of the physical deflections  $\tilde{w}$ , but also on the value of  $k$ , so that for large  $k$  the effect can be significant even for relatively small transverse deflections of the beam.

Now, consider another case when both ends of the beam are restricted to move in the longitudinal direction, and mid-plane stretching occurs. To transversely deform such a beam considerably more energy should be supplied, since bending is coupled with axial stretching of the beam. Consequently one can expect transverse deformations to be much smaller than in the previous case, so that linear measure of curvature and material stress-strain relation can be adopted, and non-linear inertia can be neglected [3]. Thus the following expression is obtained for the longitudinal force, in this case constant with respect to  $\tilde{x}$ :

$$(8) \quad \tilde{N} = \frac{1}{\int_0^{\tilde{l}} \frac{1}{EA(\tilde{x})} d\tilde{x}} \left( \eta \tilde{l} + \frac{1}{2} \int_0^{\tilde{l}} (\tilde{w}')^2 d\tilde{x} \right),$$

here  $\eta$  describes a small initial stretch of the beam, and  $\tilde{l}$  is length of the beam. Consequently, assuming the variation of spatial properties of the beam to be in accordance with (4), the following non-dimensional governing equation is obtained:

$$(9) \quad (1 + \chi_A \sin x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ (1 + \chi_I \sin x) \frac{\partial^2 w}{\partial x^2} \right] - \mu \frac{1}{\int_{\phi}^{l+\phi} \frac{1}{1 + \chi_A \sin x} dx} \left( \eta l + \frac{1}{2} \int_{\phi}^{l+\phi} \left( \frac{\partial w}{\partial x} \right)^2 dx \right) \frac{\partial^2 w}{\partial x^2} = 0.$$

Here all the involved parameters and variables are dimensionless;  $l = k\tilde{l}$  is the non-dimensional length of the beam, and  $\mu = A_0/(I_0 k^2) \gg 1$ .

### 3. Solution by the method of varying amplitudes

Solutions of Eq. (6) for the first case and Eq. (9) for the second case are sought in the form of a series:

$$(10) \quad w(x, t) = \varphi(x)e^{i\omega t} + \bar{\varphi}(x)e^{-i\omega t} + \varphi_n(x)e^{3i\omega t} + \bar{\varphi}_n(x)e^{-3i\omega t} + \dots$$

which is typical for problems involving oscillations of weakly nonlinear (homogeneous) structures with only symmetric forces being present (see, e.g. [12]). Here  $\bar{f}$  denotes the complex conjugate of  $f$ . In the present paper only those waves whose period is of the same order as or much larger than the period of modulation are considered, so that high frequency oscillations are out of the scope. Also, the nonlinearities are assumed to be weak. This permits taking into account only the first harmonic in the series (10). This simplification also agrees well with the applicability range of Bernoulli-Euler theory, which is not valid for higher modes.

Substituting (10) and (7) into (6) and balancing terms at the fundamental harmonic  $\omega$ , one obtains the following equation for the new variable  $\varphi(x)$ :

$$(11) \quad -\omega^2(1 + \chi_A \sin x)\varphi + \left[ (1 + \chi_I \sin x) \left( \varphi'' + \frac{1}{2} (2\varphi''\varphi'\bar{\varphi}' + \bar{\varphi}''(\varphi')^2) - 3\beta_n(\varphi'')^2\bar{\varphi}'' \right) \right]'' - 2\omega^2 \left[ \bar{\varphi}' \int \left( (1 + \chi_A \sin x) \int (\varphi')^2 dx \right) dx \right]' = 0,$$

where primes denote derivatives with respect to the non-dimensional spatial coordinate  $x$ . Since only waves with a period of the same order as or much larger than period of modulation are considered,  $\omega = O(1)$  (which comprises also the case  $\omega \ll 1$ ).

For the second case (9) the equation for  $\varphi(x)$  takes the form:

$$(12) \quad -\omega^2(1 + \chi_A \sin x)\varphi + \left[ (1 + \chi_I \sin x)\varphi'' \right]'' - \mu \frac{1}{\int_{\phi}^{l+\phi} \frac{1}{1 + \chi_A \sin x} dx} \left( \eta l \varphi'' + \varphi'' \int_{\phi}^{l+\phi} \varphi' \bar{\varphi}' dx + \frac{1}{2} \bar{\varphi}'' \int_{\phi}^{l+\phi} (\varphi')^2 dx \right) = 0$$

Conventional methods for analyzing spatially periodic structures, e.g. the classical Hill's method of infinite determinants [1,11] and the method of space-harmonics [6], are not applicable for the considered problem, since they are based on the Floquet theory which is valid for linear systems only. Consequently a novel approach, the method of varying amplitudes (MVA) [7,8], is employed. Here a solution of (11) (or (12)) is sought in the form of series of spatial harmonics with varying amplitudes:

$$(13) \quad \varphi(x) = b_0(x) + b_{11}(x)\exp(ix) + b_{12}(x)\exp(-ix) + b_{21}(x)\exp(2ix) + b_{22}(x)\exp(-2ix) + \dots,$$

where the complex-valued amplitudes  $b_0(x)$ ,  $b_{j1}(x)$ ,  $b_{j2}(x)$ ,  $j=1,2,\dots,m$ , are not required to vary slowly in comparison with  $\exp(ix)$ ,  $\exp(2ix)$  etc.

The shift from the original dependent variable  $\varphi(x)$  to  $2m+1$  new variables  $b_0(x)$ ,  $b_{j1}(x)$ ,  $b_{j2}(x)$  implies that  $2m+1$  equations for these variables are needed. This can be accomplished by introducing constraints on these variables in the form of  $2m$  additional equations. With the MVA

the constraints are introduced in the following way: Substitute (13) into (11), and require  $2m$  groups of terms to equal zero. The last  $2m+1$  equation will include all the remaining terms of the original equation. These  $2m$  groups of terms are proposed to be the coefficients of the lowest spatial harmonics involved, including the zeroth one, i.e.  $\exp(ix)$ ,  $\exp(-ix)$ ,  $\exp(2ix)$  etc.

Note that, by contrast to the harmonic balance method [1], truncation of the series (13) does not imply any approximations; for example with only one term taken into account,  $\varphi(x) = b_0(x)$ , the resulting equation for  $b_0(x)$  will be the same as the initial equation (11) for  $\varphi(x)$ .

Consequently, assuming the involved nonlinearities to be weak, a solution to (11) (and (12)) is obtained in the form:

$$(14) \quad w(x,t) = F(x) \exp(i(\omega t - \kappa x)) + cc,$$

where

$$(15) \quad F(x) = b_{0c} + b_{11c} \exp(ix) + b_{12c} \exp(-ix) + b_{21c} \exp(2ix) + b_{22c} \exp(-2ix) + \dots,$$

which describes a propagating (or attenuating) “compound” wave [5,6] with dimensionless frequency  $\omega$  and wavenumber  $\kappa$ . Here  $b_{0c}, b_{11c}, b_{12c}, b_{21c}, b_{22c}$  are complex constants and  $cc$  denote complex conjugate terms. The relation between  $\omega$  and  $\kappa$  defines the dispersion relation of the periodic structure considered.

## 4. Results and discussion

To simplify the analysis of the effects of nonlinearities on the dispersion relation each particular source of the nonlinearity was considered separately. First effects of nonlinear (true) curvature and nonlinear material were studied. It appears that the structure of the beam dispersion relation does not change due to these nonlinearities, i.e. the relation remains periodic with respect to  $\kappa$ , and the number of the band-gaps is the same as in the linear case. On the other hand, a shift of the band-gaps to a higher frequency range due to the nonlinear curvature is revealed. The effect of the nonlinear material is opposite, i.e. it shifts the band-gaps to a lower frequency range. So, it is possible to compensate the effect of one source of the nonlinearity by another. The effects of nonlinearities are more pronounced for higher frequencies and the corresponding band-gaps. And even weak nonlinearities can cause frequency bands-gaps, i.e. frequency ranges in which waves *cannot* propagate, to become pass-bands, i.e. frequency ranges in which waves *can* propagate, and vice versa. The width of the band-gaps, however, appears to be relatively insensitive to (weak) nonlinearities.

The effects of mid-plane stretching nonlinearity turn out to be similar to those of the nonlinear curvature: the band-gaps are shifted to a higher frequency range, while the width of the band-gaps is changed only very slightly. However, this source of nonlinearity is much stronger than the nonlinear curvature: it is pronounced even at very small values  $w \sim 10^{-2}$  of the beam transverse deflections.

Next consider the effect of nonlinear inertia, governed by the term  $-(Nw')'$  in (6), on the dispersion relation. Substituting the obtained solution  $w_l$  for the *linear* beam problem into this term, one finds that for  $\kappa = 0, \pm 0.5, \pm 1, \pm 1.5, \dots$  and  $\chi_A \neq 0$  or  $\chi_I \neq 0$  it tends to infinity for arbitrarily small values of transverse beam deflections:

$$(16) \quad \left| -\frac{\partial}{\partial x} \left( N \frac{\partial w_l}{\partial x} \right) \right|_{\kappa=0, \pm 0.5, \pm 1, \dots} \rightarrow \infty \text{ for } w_l \rightarrow 0$$



Thus the dispersion relation for the non-uniform beam should change considerably for wavenumbers close to  $\kappa = 0, \pm 0.5, \pm 1, \pm 1.5, \dots$  due to nonlinear inertia; it is exactly at these wavenumbers the frequency band-gaps arise in the linear case.

Dispersion relations, relating a certain frequency with a certain wavenumber, as well as frequency band-gaps are features of linear or weakly nonlinear waves only. For strongly nonlinear waves, comprising many components with different frequencies, these notions make no sense [5,6]. As appears, if the linear dispersion relation features band-gaps at  $\kappa = \pm 0.5$ , then the wave motion close to these wavenumbers becomes strongly nonlinear due to nonlinear inertia. This happens even for very small values of transverse beam deflections. So instead of having a band-gap at  $\kappa = \pm 0.5$ , we obtain a strongly nonlinear wave motion close to these wavenumbers, i.e. this band-gap appears to vanish due to nonlinear inertia. The band-gap at  $\kappa = 0, \pm 1$  also vanishes due to this nonlinearity. However, in this case the wave motion close to  $\kappa = 0, \pm 1$  is *weakly* nonlinear.

The effects described above are present for the non-uniform beam only; for the uniform beam the influence of nonlinear inertia on the dispersion relation is much weaker.

The obtained results clearly indicate that nonlinear inertia has a substantial impact on the non-uniform beams dispersion relation. It appears that it removes all the band-gaps by making the wave motion either strongly nonlinear ( $\kappa = \pm 0.5$ ), or weakly nonlinear and featuring no band-gaps ( $\kappa = 0, \pm 1$ ). However, the results obtained are for *continuous* modulations of the beam structural parameters. In the case of *piecewise constant* modulations the effects of nonlinear inertia can be much weaker, as is suggested by the results obtained for the uniform beam.

## 5. Conclusions

The effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations were analysed. Two different cases are considered: 1) “large” transverse deflections, where nonlinear (true) curvature, nonlinear material, and nonlinear inertia due to longitudinal motions of the beam should be taken into account, and 2) mid-plane stretching nonlinearity. As a result, several notable effects are revealed by the means of the Method of Varying Amplitudes. In particular it is shown that nonlinear inertia has a substantial impact on the dispersion relation of the non-uniform beam with continuous modulations of cross-section parameters. It appears that it removes all the band-gaps by making the wave motion either strongly nonlinear, or weakly nonlinear and featuring no band-gaps.

The work is carried out with financial support from the Danish Council for Independent Research and FP7 Marie Curie Actions – COFUND: DFF – 1337-00026.

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